

Statistical behavior of domain systems

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We study the statistical behavior of two out of equilibrium systems. The first one is a quasi-one-dimensional gas with two species of particles under the action of an external field which drives each species in opposite directions. The second one is a one-dimensional spin system with nearest-neighbor interactions also under the influence of an external driving force. Both systems show a dynamical scaling with domain formation. The statistical behavior of these domains is compared with models based on the coalescing random walk and the interacting random walk. We find that the scaling domain size distribution of the gas and the spin systems is well-fitted by the Wigner surmise, which lead us to explore a possible connection between these systems and the circular orthogonal ensemble of random matrices. However, the study of the correlation function of the domain edges shows that the statistical behavior of the domains in both gas and spin systems is not completely well-described by a circular orthogonal ensemble, nor it is by other models proposed such as the coalescing random walk and the interacting random walk. Nevertheless, we find that a simple model of independent intervals describes more closely the statistical behavior of the domains formed in these systems.

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I. INTRODUCTION

In this paper we study the statistical behavior of two non-equilibrium systems. The first one is a quasi-one-dimensional gas introduced in Ref. [1]. There, the authors studied the biased diffusion of two species in a fully periodic $2 \times L$ rectangular lattice half filled with two equal numbers of two types of particles (labeled by their charge $+$ or $-$). An infinite external field drives the two species in opposite directions along the x axis (long axis). The only interaction between particles is an excluded volume constraint, i.e., each lattice site can be occupied at most by only one particle.

The second system is a one-dimensional spin system introduced in Ref. [2], where the authors consider a chain of L Ising spins with nearest-neighbor ferromagnetic interaction J . The chain is subject to spin-exchange dynamics with a driving force E that favors motion of up spins to the right over motion to the left.

Both systems evolve with the formation of domains which grow in time. For intermediate times where the size of the domains is much smaller than the total size L of the system, the domain size distribution exhibits a dynamic scaling. In this work we will be interested in the statistical properties of these domains in the scaling regime.

For each system there are two kinds of domains. For the gas system, there are domains filled with particles and empty domains. For the spin system there are domains of up spins and domains of down spins. We are interested in the statistical properties of these domains regardless of their type (filled or empty, up or down). Let us define $P^{(n)}(S, t)$ as the probability density function that the distance between the external borders of $n+1$ consecutive domains is S at time t . Let $\langle S \rangle$ be the average of S and the relative spacing between borders is $s = S/\langle S \rangle$. The scaling probability density function is defined as

$$p^{(n)}(s) = \langle S \rangle P^{(n)}(s \langle S \rangle, t), \quad (1)$$

in particular $p^{(0)}(s)$ is the scaling nearest-neighbor edges distribution, i.e., the domain size distribution. In the scaling regime, the scaling probability density functions $p^{(n)}(s)$ do not depend on the time t .

Numerical simulation of both systems performed in [1–3] show that their scaling domain size distribution function is well-fitted by

$$p^{(0)}(s) = \frac{\pi}{2} s e^{-\pi s^2/4}. \quad (2)$$

This distribution is known in the random matrices theory as the Wigner surmise and it describes the spacing distribution between eigenvalues in the Gaussian and circular orthogonal ensembles (GOE and COE, respectively) [4]. This fact suggests a possible connection between the statistical behavior of the borders of domains in nonequilibrium systems and the eigenvalues in random matrices in a similar way as it occurs with the eigenvalues of the Gaussian orthogonal-unitary ensemble and the vicious random walk [5–7].

The main objective of this paper is to explore this possible connection between nonequilibrium domain systems and random matrices, in particular the circular orthogonal ensemble. In Secs. II and III, we summarize the main results found in Ref. [1] on the quasi-one-dimensional gas, and in Refs. [2,3] on the spin system. In Sec. IV, we recall some facts about the random matrix theory, relevant for our purposes. In Sec. V, we compare the statistical behavior of the domains for each system with the statistical behavior of the eigenvalues of random matrices from the circular orthogonal ensemble. Also, we will consider a simple model of independent intervals, which turns out to better describe the statistics of the domains in both the gas and the spin systems.

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II. QUASI-ONE-DIMENSIONAL GAS

This system was described briefly in the Introduction, more details are found in Ref. [1]. The system evolves in time according to the following dynamical rules.

(1) L particles are randomly inserted in a $2 \times L$ rectangular lattice, $\frac{L}{2}$ particles (+) and $\frac{L}{2}$ particles (−), the remaining sites are empty. Periodic boundary conditions are imposed in both directions of the lattice. Let the x axis be the long axis of length L .

(2) Two neighbor sites are chosen at random. The contents of the sites are exchanged with probability 1 if the neighbors sites are particle-hole, but if they are particle-particle the content is exchanged with probability γ . The exchanges which result in + (−) particles moving in the positive (negative) x direction are forbidden due to the action of the external field.

(3) A time unit corresponds to $2L$ attempts of exchange.

A. Qualitative results

The system was studied by computer simulation in Refs. [1,8]. We also performed several simulations of this system with a parameter $\gamma=0.1$. The following qualitative results are found from the simulations.

(1) For low γ values the system remains homogeneous, i.e., the system evolves without domain formation.

(2) For high γ values the system evolves as follows. For early times small domains form everywhere due to mutual obstruction of the opposite species. After a long time, the system settles into a nonequilibrium steady state (NESS) in which only one macroscopic domain survives, so that its length is about $\frac{L}{2}$ and it contains almost no holes. In the low density region the traveler particles leak out of the domain at one end and later rejoin it at the other end. The charge distribution of this macroscopic domain is not trivial.

(3) For intermediate times, when the average size of the clusters is much smaller than the size L of the system, the system shows dynamical scaling.

B. Quantitative results

Following Ref. [1], we can map the quasi-one-dimensional lattice into a one-dimensional lattice. This approximation described in Ref. [1] is a coarse grained description. For any configuration on the $2 \times L$ lattice we construct an effective one-dimensional one, with occupation numbers 0 or 1 on an L sites line, as follows. At each site i , we assign 0 if there are five or less particles in the ten sites around it, including the i th column of the original lattice. We assign 1 otherwise. In the coarse grained configuration a domain is a simple consecutive sequence of ones and its size is just the length of this string.

Under this coarse grained description, the authors of Ref. [1] considered the “residence distribution” of the clusters $\tilde{P}(S, t)$, defined as $\tilde{P}(S, t) = SP^{(0)}(S, t)$ in terms of the domain size distribution $P^{(0)}(S, t)$. Using the residence distribution, they found [1,8] that the average length of domains $\bar{S} = \sum_S S \tilde{P}(S, t) / \sum_S \tilde{P}(S, t)$ grows in time with an exponent of at

least 0.6. The numerical results from the simulations of Ref. [1] show that the scaling residence distribution $\tilde{p}(s) = \tilde{S} \tilde{P}(s \tilde{S}, t)$ is well-fitted by

$$\tilde{p}(s) = \frac{32}{\pi^2} s^2 e^{-4s^2/\pi}. \quad (3)$$

Remembering that $\tilde{p}(s)$ is, by definition, proportional to $sp^{(0)}(s)$, this implies that the scaling domain size distribution $p^{(0)}(s)$ is properly fitted by the Wigner surmise for the orthogonal ensembles of random matrices, Eq. (2). We performed our own simulations to confirm this, the results will be shown in Sec. V.

In Ref. [1], the authors proposed an approximate model to describe this system, in the coarse grained description, using the coalescing random walk (CRW) where the particles execute independent random walks, suffering a fusion reaction ($A + A \rightarrow A$) when two particles meet. This approximation is useful because the full hierarchy of correlations for this diffusion-limited reaction has been solved, see Refs. [9–12]. This approximation is done under the following assumptions.

(1) The number of domains is equal to the number of walkers in the CRW.

(2) The walkers describe random walks over a one-dimensional ring with N sites, in that way, N is equal to the sum of the length of the domains.

(3) The distances between walkers are equal to the length of the domains.

(4) When two walkers meet they coalesce, representing the disappearance of a domain.

(5) The traveling particles of the original model are neglected in the CRW description.

Under this approximation the spacing distribution is the same found as in the coarse grained description, but the average length of domains grow in time with an erroneous exponent of 0.5.

In order to solve this problem, in Ref. [1], the authors introduced the model of the interacting random walkers (IRW), where neighboring walkers experience stronger attraction with decreasing separation. The probability of a walker moving to the right is

$$q_r = q_s \left[1 + \left(\frac{0.06 \langle S \rangle}{l_r} \right)^2 \right], \quad (4)$$

where l_r is the distance to its right nearest neighbor and q_s is the probability of staying. In the same way, the probability of moving to the left is

$$q_l = q_s \left[1 + \left(\frac{0.06 \langle S \rangle}{l_l} \right)^2 \right], \quad (5)$$

where l_l is the distance to its left nearest neighbor and $q_s = 1 - q_r - q_l$. The factor $0.06 \langle S \rangle$ is chosen in Ref. [1] in order to give the best fit to the simulation data and the dependence with l^{-2} is justified in Ref. [1]. In this model, \bar{S} grows with an exponent of 0.6, see Ref. [1]. The spacing distribution is still given by Eq. (2).

We performed a few simulations of this interacting random walk changing the constant 0.06 in Eqs. (4) and (5). This has the effect of changing the growth exponent of the domains, however, the scaling domain size distribution $p^{(0)}(s)$ remains unchanged, it is still given by the Wigner surmise (2).

III. ONE-DIMENSIONAL SPIN SYSTEM

In the Introduction we briefly presented this system, for more details see Refs. [2,3]. The lattice has a length N with $N\mu$ spins up (“+”) and $N(1-\mu)$ spins down (“−”) with $0 < \mu < 1$. Periodic boundary conditions are imposed. The spin-flip events are

- (1) $++-- \leftrightarrow +-+-$ $\Delta = 4J - E$.
- (2) $--++ \leftrightarrow -+-+$ $\Delta = 4J + E$.
- (3) $+-+- \leftrightarrow +-+-$ $\Delta = -E$.
- (4) $-+-+ \leftrightarrow -+-+$ $\Delta = -E$, where the transition probability rate for a process from left to right is proportional to $\frac{1}{2}[1 - \tanh(\frac{\Delta}{2T})]$. The constant J is the nearest-neighbor coupling between spins, E is the energy associated to an external field which drives the up (“+”) spins to the right and the down (“−”) spins to the left, and T the thermal energy (temperature times Boltzmann constant).

As explained in [2], the microscopic dynamics of the lattice of spins may be mapped onto one for an array domain dynamics, which provides a good approximation in the regime $T \ll E \ll J$. In this approximation, domains of up spins move spontaneously to the right, and domains of down spins move to the left. The rates for such processes are independent of the domain size. The algorithm used for the numerical simulation is the following.

- (1) Set up a random array of alternating down and up spins, with μN spins up and $(1-\mu)N$ spins down.
- (2) Choose a domain at random.
- (3) If the domain is down, move it to the left (i.e., reduce the size of its left neighbor by one and increase its right neighbor by one), otherwise move it to the right.
- (4) If one of the neighbor domains is zero size, then the domain is removed.
- (5) Update the clock by $1/\text{number of domains}$.
- (6) Repeat steps 2–6.

This simplified system allows much better statistics than it would be possible with the true microscopic system [2], and its simulation is more easy to implement and it reproduces the same asymptotic behavior.

A. Qualitative results

From the simulation, the following qualitative results are observed.

- (1) For early times, little domains form everywhere.
- (2) At intermediate times, some domains disappear while other domains grow. In this time regime, the system shows dynamical scaling.
- (3) For later times, only two macroscopic domains remain, which move in opposite directions. The system falls in a nonequilibrium steady state.

B. Quantitative results

In Ref. [2], the authors derive an analytical solution in the case $\mu \rightarrow 0$, nevertheless we can derive another exact solution for the dynamical domain model with $\mu = 0.5$. Let $P^{(0)}(n, m, t)$ be the frequency of finding a domain of spin up with length n and another domain of spin down with length m at time t . The master equation for $P^{(0)}(n, m, t)$ is

$$\begin{aligned} \frac{dP^{(0)}(n, m, t)}{dt} = & P^{(0)}(n+1, m, t) + P^{(0)}(n-1, m, t) \\ & - 2P^{(0)}(n, m, t) + P^{(0)}(n, m+1, t) \\ & + P^{(0)}(n, m-1, t) - 2P^{(0)}(n, m, t). \end{aligned} \quad (6)$$

Let Δx be the distance between nearest-neighbor sites of the lattice. In the continuous limit, $\Delta x \rightarrow 0$, the above probability is a function of $S = n\Delta x$, $R = m\Delta x$. It satisfies the master equation

$$\frac{dP^{(0)}(S, R, t)}{dt} = \frac{\partial^2 P^{(0)}(S, R, t)}{\partial S^2} + \frac{\partial^2 P^{(0)}(S, R, t)}{\partial R^2}. \quad (7)$$

If at $t=0$ all domains have length S_0 , the above equation is subject to the initial condition $P^{(0)}(S, R, 0) = \delta(S - S_0)\delta(R - S_0)$ and the boundary conditions $P^{(0)}(0, R, t) = 0$ and $P^{(0)}(S, 0, t) = 0$. The appropriate scaling solution for this differential equation is

$$p^{(0)}(s, r) = \frac{\pi^2}{4} s r e^{-\pi s^2/4} e^{-\pi r^2/4}, \quad (8)$$

as a consequence, the length of domains of spin up and of domains of spin down have the same probability distribution function found in the quasi-one-dimensional gas. The above equation can be written as the product of two distributions with the form of the Wigner surmise (2).

The fact that the domain size distribution $p^{(0)}(s)$ for a domain of up or down spins is well-described by the Wigner surmise (2) was also confirmed in Refs. [2,3] from the numerical simulation results.

IV. RANDOM MATRICES

Historically, the initial motivation to introduce random matrices was to study the statistics of the energy levels of some quantum systems, in nuclear physics initially, then for quantum systems when their classical counterpart is chaotic, for a review see Ref. [4]. Some local properties, such as the spacing between consecutive high energy levels are well-described by the random matrix theory. For quantum chaotic systems with time-reversal invariance and integral total angular momentum, the high energy levels, for a given value of the spin and the parity, have spacing distributions and correlations that are well-described by the Gaussian orthogonal ensemble (GOE) of random matrices, when the energy levels are rescaled such that the average interspacing is unity. For large matrices, the (properly rescaled) statistics of the eigenvalues of the GOE in the bulk is the same as the one for the circular orthogonal ensemble (COE). In this work we preferred to work with the circular orthogonal ensemble rather

than the Gaussian orthogonal ensemble since the former one is periodic and that way we avoid boundary condition problems (in the GOE the statistics of the eigenvalues near the edge are different than in the bulk).

The COE is a statistical ensemble of $N \times N$ unitary symmetric matrices S invariant under the transformations $S \rightarrow W^T S W$, where W is any $N \times N$ unitary matrix. The eigenvalues of S are of the form $e^{i\theta_k}$, with $k=1, \dots, N$. The probability density function for the eigenvalues is

$$P_{N\beta}(\theta_1, \dots, \theta_N) = \frac{1}{Z_{N\beta}} \prod_{1 \leq k < j \leq N} |e^{i\theta_k} - e^{i\theta_j}|^\beta \quad (9)$$

with $\beta=1$. The partition function is

$$Z_{N\beta} = (2\pi)^N \Gamma(1 + \beta N/2) / [\Gamma(1 + \beta/2)]^N$$

[4]. It should be noted that the eigenvalue probability density function of the COE is equal to the Boltzmann factor of an equilibrium system of particles moving on a circle and interacting through a log-potential, the two-dimensional Coulomb potential:

$$P_{N\beta}(\theta_1, \dots, \theta_N) = \frac{1}{Z_{N\beta}} e^{-\beta V_N(\theta_1, \dots, \theta_N)} \quad (10)$$

with

$$V_N(\theta_1, \dots, \theta_N) = \sum_{1 \leq k < j \leq N} -\ln |e^{i\theta_k} - e^{i\theta_j}|. \quad (11)$$

Up to an additive constant, the log-potential is scale invariant, thus in these log-gases the density plays only the trivial role of fixing the length scale of the problem. In the other systems we study here, it is important to rescale the lengths in order to have the interparticle average spacing equal to one. For the nonequilibrium systems, this has been possible due to the fact that they exhibit a scaling regime.

The COE is a homogeneous system, the density of eigenvalues is constant on the circle, equal to $2\pi/N$. It is convenient to rescale the lengths such that the average spacing between eigenvalues is unity. Let $x_k = N\theta_k/(2\pi)$ be the rescaled eigenvalues (note a small abuse of language here, since the eigenvalues are really $e^{i\theta_k}$ not θ_k). Analytic expressions for the eigenvalue spacing distributions are known, however, they are complicated, see Ref. [4]. It turns out that a very accurate approximate expression for the nearest-neighbor spacing distribution $p^{(0)}(s)$ is given by the Wigner surmise (2). In the limit of large matrices, $N \rightarrow \infty$, the correlation function between two (rescaled) eigenvalues x and x' is given by

$$g(r) = \left[\int_r^\infty s(r') dr' \right] \frac{ds}{dr}(r) + [s(r)]^2 \quad (12)$$

with $r = |x - x'|$ and $s(r) = \sin(\pi r)/(\pi r)$.

V. RESULTS AND DISCUSSION

A. Nearest-neighbor spacing distribution and Wigner surmise

We performed simulations for the gas, the spin, the CRW, the IRW, and the circular orthogonal ensemble (COE) of ran-

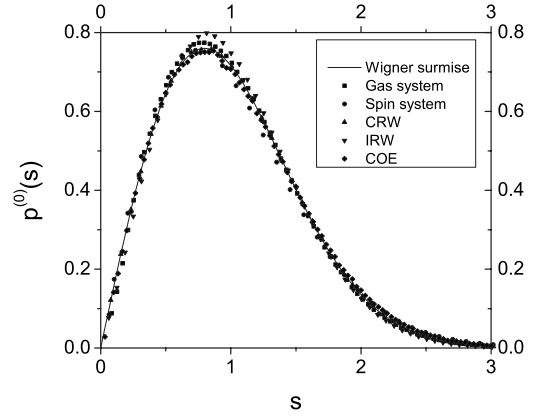


FIG. 1. Nearest-neighbor spacing distributions for the gas, spin, IRW, CRW, COE, and Wigner surmise.

dom matrices with the following parameters. For the gas, the spin, and the CRW systems, we used a lattice with 1000 sites, and for the IRW the lattice had 500 sites. The random matrix simulations were performed with 200×200 matrices. For the nonequilibrium systems, the simulations were carried out long enough to reach the scaling regime. The initial density of particles in the CRW and the IRW was $1/2$ and $2/3$, respectively. In the scaling regime, we build histograms for the domain sizes (i.e., the nearest neighbor spacings for the CRW, IRW, and COE). The data to build the histograms was taken at three different times for each system in order to verify the existence of a proper scaling regime. These times were $T=1000, 1500$, and 2000 for the gas system; $T=10, 18$, and 34 for the spin system (the time unit of each system was defined in the previous sections); $T=50, 100$, and 200 for the CRW, and $T=100, 150$, and 200 for the IRW. For the IRW and the CRW systems, the Monte Carlo time unit corresponds to a trial to move all the particles of the system. To have appropriate statistics, we performed 20 000 simulations (realizations) of the gas, the spin, and the COE systems, and 50 000 realizations of the CRW and the IRW systems.

Figure 1 shows the scaling domain size distribution $p^{(0)}(s)$ for the gas and the spin systems, obtained from the simulations. It is compared with the scaling nearest-neighbor distribution of random walkers in the IRW and the CRW, and the scaled nearest-neighbor distribution of eigenvalues of the circular orthogonal ensemble, which we also obtained from the simulation of these systems. We also present the Wigner surmise, Eq. (2), which is known to numerically reproduce accurately the nearest-neighbor distribution of eigenvalues of the circular orthogonal ensemble (it is exact for 2×2 matrices).

As we can see in Fig. 1, the Wigner surmise reproduces correctly the nearest-neighbor distribution $p^{(0)}(s)$ for the gas, the spin, the IRW, the CRW, and the COE systems. This suggests a possible connection between the random matrices and the out of equilibrium systems.

Motivated by this possible link between the nonequilibrium statistical systems and the random matrix theory (i.e., the equilibrium log-gas system), we explore the possibility if not only the nearest-neighbor distribution is the same for all these systems, but also the other spacing distributions

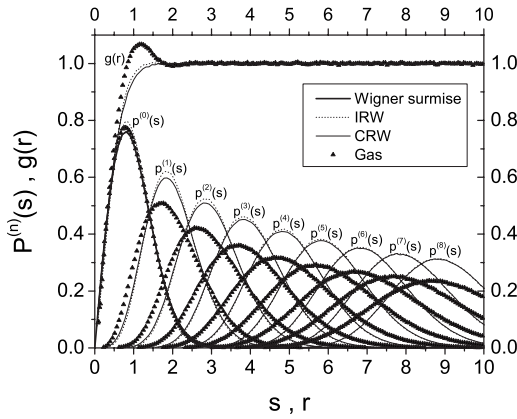


FIG. 2. Spacing distributions and the pair correlation function in the gas system, IRW, and CRW for a lattice with $N=500$ and 1000 sites, respectively.

$p^{(n)}(s)$ for $n > 0$ and the pair correlation function $g(r)$. The approach here is to consider the edges of the domains in the gas and spin system as fundamental entities, and to study their spacing distributions and correlation function, in the scaling regime. With this study we wish to know if the similitude between these systems goes beyond the nearest-neighbor distribution, or if it is simply a coincidence that the nearest-neighbor distribution is the same for all systems. This also can give an indication about how much information on each system is contained in the nearest-neighbor distribution.

From our simulations, we computed the other spacing distributions $p^{(n)}(s)$ and the correlation function $g(r)$ of all these systems in order to compare them. The results are presented in the following sections.

B. Quasi-one-dimensional gas and its approximate models: IRW and CRW

In Ref. [1], the domain size distribution of the quasi-one-dimensional gas was described by the approximate models of random walkers, the IRW and CRW. We now compare the other spacing distribution functions and the correlation function. The results are shown in Fig. 2. The CRW and IRW reproduce correctly the nearest-neighbor distribution $p^{(0)}(s)$; but, both the IRW and the CRW fail describing the other spacing distribution functions $p^{(n)}(s)$ with $n > 0$, and, as a consequence, they also fail describing the two-point correlation function $g(r)$, since the correlation function can be expressed as

$$g(r) = \sum_{n=0}^{\infty} p^{(n)}(r). \quad (13)$$

We notice in particular the fact that the correlation function of the gas system shows an oscillation around $r=1$ (and probably more for higher values of r). On the contrary, the CRW and IRW models show no oscillation in the correlation function $g(r)$. For these systems $g(r)$ seems to be a monotonic function.

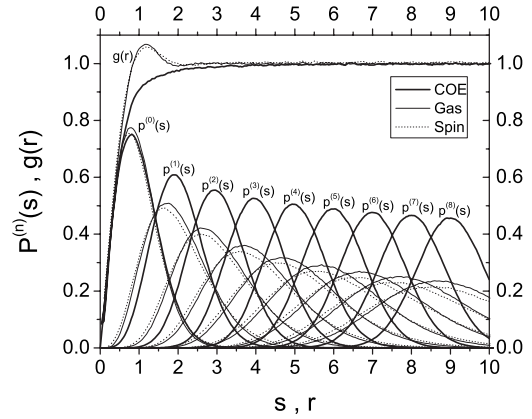


FIG. 3. Comparison between the gas and spin systems ($\mu=0.5$) and COE for a lattice with $N=1000$ sites. In the COE we took 20 000 matrices of 200×200 .

The spacing distributions $p^{(n)}(s)$ for $n \geq 1$ of the IRW and CRW are somehow similar, but show higher maxima values than those of the gas system. Furthermore, the maxima of the IRW and the CRW are located at higher values of s than the corresponding ones of the gas system.

We conclude that the CRW and IRW systems are not statistically equivalent to the quasi-one-dimensional gas, although these models were proposed in Ref. [1] to describe (approximately) the quasi-one-dimensional gas.

C. Gas and spin systems and the circular orthogonal random matrix ensemble (COE)

Now, in Fig. 3, we compare the spacing distributions $p^{(n)}(s)$ and the correlation function $g(r)$ of both the gas and spin systems with the ones for the random matrix COE. The correlation functions $g(r)$ of the gas and the spin systems seem to be very similar, almost identical, with only small differences of a few percent in relative difference. They both exhibit the oscillation near $r=1$ mentioned above. On the contrary, the correlation function for the COE does not show any oscillation, and it is very different from the correlation function of the gas and spin systems.

The spacing distributions $p^{(n)}(s)$ are somehow similar for the gas and the spin systems, with some small differences between them; but, in any case, they differ much from the ones of the COE. The distributions $p^{(n)}(s)$, for $n \geq 1$, are more “disperse” in the gas and spin systems than the circular orthogonal ensemble, i.e., they have a width at half height larger than the ones for the COE. Also, the maxima for each distribution of the gas and spin systems are located at smaller values of s than for the COE. The maxima values are also smaller for the gas and spin systems than for the COE.

It is possible to conclude that the gas and spin systems have a similar statistical behavior but this behavior is different from the COE. They only coincide in the nearest-neighbor distribution.

In the previous section, we have shown that the CRW and the IRW systems do not describe properly the statistics of the spacing distributions of the gas system; but do they have a similar behavior as the COE? In Fig. 4, we compare the

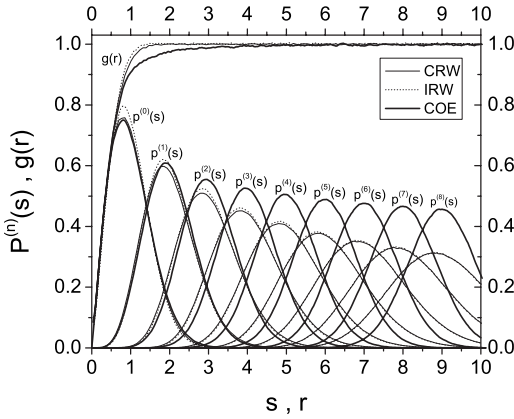


FIG. 4. Comparison between the CRW, IRW, and COE for a lattice with $N=1000$ sites and in COE we take 20 000 matrices of 200×200 .

correlation functions and spacing distribution functions of both the CRW and IRW systems with COE.

The correlation function $g(r)$ of the CRW and IRW differs from the one of the COE. Although the three correlation functions are monotonous (they have no oscillations), around $1 \leq r \leq 2$, the correlation function of the COE is smaller than the one of the IRW and the CRW. The first two distribution functions $p^{(0)}(s)$ and $p^{(1)}(s)$ of the three systems, IRW, CRW, and COE, are very similar. However, from $n \geq 2$, the COE spacing distributions $p^{(n)}(s)$ start to differ from the corresponding ones of the CRW and the IRW. The spacing distributions of the CRW and the IRW seem more “disperse” (as defined above) than the ones of the COE. The maxima values for the spacing distributions in the COE are larger than for the IRW and CRW. Also, these maxima are located at smaller values of s for the IRW and CRW than for the COE.

We conclude that the CRW and the IRW systems are not statistically equivalent to the COE. On the other hand the CRW and the IRW show very similar spacing distribution functions and correlation functions. Thus another important conclusion is that the interaction proposed in Ref. [1] for the IRW does not change considerably the statistical behavior of the domains of the system from the CRW in the scaling regime. The interaction between neighbors, Eqs. (4) and (5), changes the growth exponent of the domains; but, once rescaled, the statistics of the walker spacing distributions and correlations of the CRW and IRW are very similar.

D. Independent interval approximation (IIA)

The results presented above show that the statistics of the edges of the domains of the out of equilibrium gas and spin systems, properly rescaled in the scaling regime, are not described by the COE, and thus are not described by an equilibrium statistical system of particles with a log-potential interaction. Furthermore, for the gas systems, the approximate models of CRW and IRW do not reproduce correctly higher order spacing distributions and the correlation function. In this section, we propose a model that describes better the statistics of the domains of the out of equilibrium systems in the scaling regime when properly rescaled.

We have found numerical evidence that suggests that the statistical behavior of the domains of both the gas and spin systems can be fairly well-described by the independent interval approximation (IIA), where the correlation between domains is neglected. With this simplification, it is possible to calculate the distributions $p^{(n)}(s)$, for any $n \geq 1$, see Refs. [13–17]. For example, for $n=1$, the probability to have two domain edges separated by a distance s , knowing that there is one edge in between at a distance r from the first edge is $p^{(0)}(r)p^{(0)}(s-r)$, if the domains are considered independent. Therefore

$$p^{(1)}(s) = \int_0^s dr p^{(0)}(r) p^{(0)}(s-r). \quad (14)$$

It is useful to introduce here the Laplace transform of the distribution functions

$$\tilde{p}^{(n)}(l) = \int_0^\infty p^{(n)}(s) e^{-ls} ds. \quad (15)$$

Taking the Laplace transform of Eq. (14), we have

$$\tilde{p}^{(1)}(l) = [\tilde{p}^{(0)}(l)]^2. \quad (16)$$

If the Wigner surmise (2) is assumed for $p^{(0)}(s)$, its Laplace transform is

$$\tilde{p}^{(0)}(l) = 1 - l e^{l^2/\pi} \operatorname{erfc}\left(\frac{l}{\pi}\right), \quad (17)$$

where $\operatorname{erfc}(z) = (2/\sqrt{\pi}) \int_z^\infty e^{-t^2} dt$ is the complementary Gaussian error function. Using the inverse Laplace transform of expression (16) or from a direct calculation of Eq. (14), we find

$$p^{(1)}(s) = \frac{\pi}{16} e^{-\pi s^2/4} \left[4s + \sqrt{2} e^{\pi s^2/8} (-4 + \pi s^2) \operatorname{erf}\left(\frac{s}{2} \sqrt{\frac{\pi}{2}}\right) \right], \quad (18)$$

with $\operatorname{erf}(z) = 1 - \operatorname{erfc}(z)$.

More generally, under this approximation, the Laplace transform of $p^{(n)}(s)$ is simply

$$\tilde{p}^{(n)}(l) = [\tilde{p}^{(0)}(l)]^n = \left[1 - l e^{l^2/\pi} \operatorname{erfc}\left(\frac{l}{\pi}\right) \right]^n. \quad (19)$$

Since the pair correlation function can be obtained from Eq. (13), then $g(r)$ is given by the sum over convolutions of $p^{(0)}(r)$, see Ref. [13]. In terms of the Laplace transform \tilde{g} of the correlation function, we have

$$\tilde{g}(l) = \sum_{k=1}^{\infty} [\tilde{p}^{(0)}(l)]^k = \frac{\tilde{p}^{(0)}(l)}{1 - \tilde{p}^{(0)}(l)}, \quad (20)$$

with $\tilde{p}^{(0)}(l)$ given by Eq. (17). We have

$$\tilde{g}(l) = \frac{1 - le^{l^2/\pi} \operatorname{erfc}\left(\frac{l}{\sqrt{\pi}}\right)}{le^{l^2/\pi} \operatorname{erfc}\left(\frac{l}{\sqrt{\pi}}\right)}. \quad (21)$$

The inverse Laplace transform of the above expression can be computed numerically to obtain $g(r)$.

In this independent interval approximation, the joint probability density function to find borders in positions x_1, x_2, \dots, x_N in a line of length L is given by

$$P_N(x_1, \dots, x_N) = \frac{1}{Z_N} p^{(0)}(x_2 - x_1) \cdots p^{(0)}(x_N - x_{N-1}) p^{(0)}(x_1 + L - x_N), \quad (22)$$

in compact form

$$P_N(x_1, \dots, x_N) = \frac{1}{Z_N} \prod_{i=1}^N p^{(0)}(x_{i+1} - x_i), \quad (23)$$

where we have considered periodic boundary conditions and we defined $x_{N+1} = x_1 + L$. The partition function Z_N is the normalization constant

$$Z_N = \int_{x_1 < x_2 < \dots < x_N < x_1 + L} dx_1 \cdots dx_N \prod_{i=1}^N p^{(0)}(x_{i+1} - x_i). \quad (24)$$

Comparing Eq. (23) with a Boltzmann factor with an inverse temperature $\beta=1$, we see that, under the independent interval approximation, the statistics of the domain edges is equivalent to a system with N particles which interact according to the potential

$$V_N(x_1, \dots, x_N) = \sum_{i=1}^N \left[\frac{\pi}{4} (x_{i+1} - x_i)^2 - \ln \left(\frac{\pi}{2} (x_{i+1} - x_i) \right) \right]. \quad (25)$$

Thus it is equivalent to a statistical equilibrium system of particles on a circle interacting through a nearest-neighbor pair potential. The partition function and correlations can be computed by means of the Laplace transform as explained in general for this kind of system in Ref. [18]. We already computed the correlation function $g(r)$ in Eqs. (20) and (21). The partition function can be obtained as follows. Doing a change of variable $x_i \rightarrow x_i - x_1$ in Eq. (24) we have

$$Z_N = L \int_{0 < x_2 < \dots < x_N < L} dx_2 \cdots dx_N p^{(0)}(x_2) \left(\prod_{i=3}^N p^{(0)}(x_i - x_{i-1}) \right) p^{(0)}(L - x_N). \quad (26)$$

This is an N -fold convolution product of $p^{(0)}(x)$. Thus the Laplace transform of the partition function is

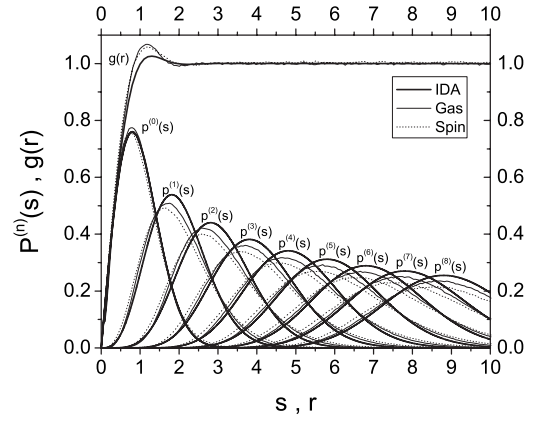


FIG. 5. Comparison between the gas and spin systems and IIA for a lattice with $N=1000$ sites.

$$\int_0^\infty e^{-lL} \frac{Z_N}{L} dL = [\tilde{p}^{(0)}(l)]^N = \left[1 - le^{l^2/\pi} \operatorname{erfc}\left(\frac{l}{\sqrt{\pi}}\right) \right]^N. \quad (27)$$

We recall that the circular orthogonal ensemble is equivalent to a system of particles on a circle interacting through a log-potential, see Eq. (11). From Eqs. (25) and (11) the difference is clear between the independent domain approximation and the circular orthogonal ensemble, in the first system there are only nearest-neighbor interactions, while in the second one every single particle interacts with all the other particles, the pair potential is not restricted to nearest neighbors. Additionally, the functional form of both potentials is different.

In Fig. 5, we compare the gas and spin system correlation functions and spacing distributions with the theoretical predictions from the independent interval approximation. We notice that this approximation reproduces more closely the spacing distributions and the correlation function. The IIA correlation function has an oscillatory behavior near $r=1$ as it occurs in the gas and spin systems. Although the predictions are not identical, the IIA gives a good approximation for the correlation function and spacing distributions with an error of a few percent (5%) deviation from the numerical results for the gas and the spin systems.

The fact that the independent interval approximation reproduces much better the correlation function and spacing distributions of the gas and the spin systems than the other approaches considered (CRW, IRW, and COE), suggests that in the gas and spin systems the domains are not strongly correlated.

VI. CONCLUSION

We studied two out of equilibrium systems, the gas and spin systems, which present a formation of growing domains. We studied the statistical properties of the edges of the domains in the scaling regime and compared them with several other models. Since in the scaling regime the statistical properties of the edges of the domains are time independent, when properly rescaled, we tried to find if there

exists an equilibrium statistical system which reproduces these statistics. A first candidate was the circular orthogonal ensemble of random matrices (COE), since the scaling size distribution of the domains satisfies the Wigner surmise.

Studying the other spacing distributions $p^{(n)}(s)$ and the correlation function, we found that only the nearest-neighbor distribution ($n=0$) coincides for the gas system, the spin system, the COE, and the models of coalescing and interacting random walkers (CRW and IRW) which were introduced in Ref. [1] to explain the properties of the gas system. The distribution functions $p^{(n)}(s)$ for $n \geq 1$ and the correlation function are different for each of these systems. The nearest-neighbor distribution for the coalescing random walk is obtained by solving the diffusion equation with an absorbing condition in $s=0$ [1,2], and the scaling solution turns out to be given by Eq. (2). As a side note, it is interesting to notice that, up to a normalization, this is the wave function of the first excited state of the harmonic oscillator, an analogy that has been used in the context of persistence, see Refs. [19,20]. Our work shows that it is merely a coincidence that this nearest-neighbor distribution turns out to be the Wigner distribution from the random matrices theory.

We found a model which provides a better approximation for the gas and the spin system domain statistics. This model is the independent interval approximation, even though in this approximation the correlation between domains is not taken into account as it occurs in an interacting and coalesc-

ing random walk. The independent interval approximation is statistically equivalent to an equilibrium system of particles with only nearest-neighbor interactions.

The fact that the Wigner surmise, Eq. (2), reproduces correctly the nearest-neighbor spacing distribution of all the different systems we considered here, but that the other spacing distributions differ from system to system, seems to show that the nearest-neighbor distribution has some kind of universality while the other distributions do not. However, it can also be interpreted as an indication that the nearest-neighbor distribution does not contain enough information about the statistics of the system: several different systems could share the same nearest-neighbor distribution, while the finer details and differences between them are contained in the correlation function and the other spacing distributions. This is the most important result of this paper because in many cases complex systems are mapped onto more simple systems with the only criteria of the nearest-neighbor distribution similitude.

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